Matrices

1. If $\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 7 \\ 1 & 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 1 & 0 \\ 6 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$, find $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{A}$, $\mathbf{AB}$ and $\mathbf{BA}$.

2. Evaluate

(i) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 1 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \\ -1 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3. Give $\phi(\lambda) = -2 - 5\lambda + 3\lambda^2$ and $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, prove that $\phi(\mathbf{A}) = \begin{pmatrix} 14 & 2 \\ 3 & 14 \end{pmatrix}$.

4. Show that $\mathbf{A}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ then $\mathbf{A}(\theta)\,\mathbf{A}(\phi) = \mathbf{A}(\theta+\phi)$, and give a geometrical interpretation of this result.

5. If $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, prove that $\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}n(n-1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$.

6. Prove that $\mathbf{P}^2 = \mathbf{P}$ if $\mathbf{P}$ is :

(i) $\begin{pmatrix} 25 & -20 \\ 30 & -24 \end{pmatrix}$ (ii) $\begin{pmatrix} -26 & -18 & -27 \\ 21 & 15 & 21 \\ 12 & 8 & 13 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

7. If $\mathbf{K} = \begin{pmatrix} \cos\theta & a\sin\theta \\ -\frac{1}{a}\sin\theta & \cos\theta \end{pmatrix}$, prove by induction that $\mathbf{K}^n = \begin{pmatrix} \cos n\theta & a\sin n\theta \\ -\frac{1}{a}\sin n\theta & \cos n\theta \end{pmatrix}$.

8. Comment on the following argument :

$$\begin{cases} 3x+4y=1 \\ 6x+8y=3 \end{cases} \Rightarrow \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 & -4 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$\therefore \quad 0 = -4 = 3$.

9. Find all the 2×2 matrices $\mathbf{X}$ which satisfy the equation $x^2 - 4x + 3I = 0$, $\mathbf{I}$ being the unit matrix and $\mathbf{0}$ the null matrix.

10. If A and B are square matrices satisfying the equation $\mathbf{A}^2 = \mathbf{B}$, show that $\mathbf{A}$ and $\mathbf{B}$ commute.

Hence find matrices $\mathbf{A}$ such that $\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

11. Find $\mathbf{A}^{-1}$ if $\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 3 & 1 & 5 \\ 3 & 2 & 3 \end{pmatrix}$.

12. In Number 11, find $(\mathbf{A} + \mathbf{A})^{-1}$ and $\mathbf{A}^{-1} + \mathbf{A}^{-1}$. Are they equal ?

13. Show that if $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices of the same order, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

14. The 2×2 matrices $\sigma_1, \sigma_2, \sigma_3$, I are defined by :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{where} \quad i^2 = -1 .$$

Prove that:

(i) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbf{I}$

(ii) $\sigma_1 \sigma_2 = \sigma_2 \sigma_1 = i\,\sigma_3$, $\sigma_2 \sigma_3 = -\sigma_3\sigma_2 = i\,\sigma_1$, $\sigma_3 \sigma_1 = -\sigma_1\sigma_3 = i\,\sigma_2$

Show also that, if θ is a real number, $\mathbf{I} + \sum_{n=1}^{\infty} \frac{(i\sigma_1\theta)^n}{n!} = \mathbf{I}\cos\theta + i\sigma_1 \sin\theta$.

15. A matrix $\mathbf{A}$ is said to be idempotent if $\mathbf{A}^2 = \mathbf{A}$. Prove that, if $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{A}$ is any $n \times n$ idempotent matrix, then

(i) $(\mathbf{I}_n - \mathbf{A})$ is idempotent .

(ii) If $\mathbf{A}$ is non-singular, then $\mathbf{A} = \mathbf{I}_n$.

Show that every 2×2 idempotent matrix, different from $\mathbf{I}$ and $\mathbf{0}$, is of the form $\begin{pmatrix} r & q \\ r & 1-p \end{pmatrix}$,

where p, q, r satisfy $p(1 - p) = qr$.

16. If $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any 2×2 matrix with complex elements, write down $\mathbf{X}^2$ and show that if $\mathbf{X}^2 + p\mathbf{X} + q\mathbf{I} = 0$, where p, q are complex, then either

(i) $\mathbf{X}$ has trace $-p$ and determinant q,

or (ii) $\mathbf{X} = \lambda I$, where λ is a root of the equation $x^2 + px + q = 0$.

(The trace of a square matrix is the sum of the elements in its leading diagonal. $\mathbf{I}$ is the unit matrix and 0 is the zero matrix.)

17. If $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}$, find a matrix $\mathbf{B}$, of the form $\begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix}$, such that $\mathbf{B}^{-1}\mathbf{AB}$ is a diagonal matrix.

Hence, or otherwise, express A^n , where n is a positive integer, in the form of a 2×2 matrix.

18. If $\mathbf{S} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ and $\mathbf{A}$ is a 2×2 symmetric matrix with element a_{ik} ,

show that $\mathbf{B} = \mathbf{SAS}'$ is diagonal if $\tan 2\alpha = \frac{2a_{12}}{a_{11} - a_{22}}$. Verify that Tr $\mathbf{B}$ = Tr $\mathbf{A}$.

19. If λ is a constant and **x** a vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ which make $\mathbf{Ax} = \lambda\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

show that $\lambda^3 - \lambda^2(a_{11} + a_{22} + a_{33}) + \lambda(\mathbf{A}_{11} + \mathbf{A}_{22} + \mathbf{A}_{33}) - |\mathbf{A}| = 0$,

where $\mathbf{A}_{rs}$ is the co-factor of a_{rs} in $|\mathbf{A}|$.

Find the values of λ is $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and find the vector **x** corresponding to the largest value of λ.

20. If **A** is the matrix $\begin{pmatrix} r & s \\ s & r \end{pmatrix}$, where $s \neq 0$, find numbers α and δ such that $\mathbf{A}^2 = \alpha\mathbf{A} - \delta\mathbf{I}$, where **I** is the 2×2 unit matrix.

Prove by induction on n, that $2s\,\mathbf{A}^n = \lambda_n\,\mathbf{A} - \mu_n\,\mathbf{I}$ (n = 2, 3, …)

where $\lambda_n = (r + s)^n - (r - s)^n$

and $\mu_n = (r - s)\,(r + s)^n - (r + s)\,(r - s)^n$

21. Prove that if **A** is skew-symmetric (i.e. $\mathbf{A} = -\mathbf{A}^T$), then $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$ is orthogonal. (assuming that $\mathbf{I} + \mathbf{A}$ is non-singular.)

22. If $\mathbf{S} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, show that **S** is an orthogonal matrix ($\mathbf{SS'} = \mathbf{I}$). Hence show that if

$\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ then **SPS'** is a diagonal matrix.

23. Verify that $\mathbf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ is an unitary matrix ($\overline{A}'A = I$).

24. If $\mathbf{X} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

prove that $\mathbf{P}^{-1}\mathbf{XP}$ is a diagonal matrix and that **X** satisfies the equation $\mathbf{X}^3 - 2\mathbf{X}^2 - \mathbf{X} + 2\mathbf{I} = 0$, where **I** is the unit matrix.

25. Find $\mathbf{A}^{-1}$ when $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ -1 & -1 & 3 \\ 5 & 1 & 1 \end{pmatrix}$. Hence solve the system of equations : $\begin{cases} x_1 + 3x_2 + 4x_3 = h_1 \\ -x_1 - x_2 + 3x_3 = h_2 \\ 5x_1 + x_2 + x_3 = h_3 \end{cases}$.

26. Solve **(i)** $\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 - x_2 + 2x_3 = 9 \\ 2x_1 + \quad x_3 = 6 \end{cases}$ **(ii)** $\begin{cases} x_1 + x_2 + x_3 + x_4 = 4 \\ x_1 - x_2 - 2x_3 + x_4 = 6 \end{cases}$ **(iii)** $\begin{cases} 2x_1 = 2 \\ x_1 + 3x_2 = 1 \\ 4x_1 + 2x_2 = 3 \end{cases}$.

27. Solve **(ii)** $\begin{cases} x_1 + 3x_2 = 4 \\ 2x_1 - 2x_2 = 6 \end{cases}$ **(ii)** $\begin{cases} x_1 - 4x_2 = 2 \\ 2x_1 + x_2 = 1 \end{cases}$

(iii) $\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 \quad + x_3 = 1 \\ x_2 \quad - x_3 = 3 \end{cases}$ **(iv)** $\begin{cases} 2x_1 + 5x_2 + 3x_3 = 1 \\ 3x_1 + x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + x_3 = 0 \end{cases}$

28. Find a non-trivial solution for the equations

$\begin{cases} \dfrac{x_1}{a+\lambda_1} + \dfrac{x_2}{a+\lambda_2} + \dfrac{x_3}{a+\lambda_3} = 0 \\ \dfrac{x_1}{b+\lambda_1} + \dfrac{x_2}{b+\lambda_2} + \dfrac{x_3}{b+\lambda_3} = 0 \end{cases}$ where none of the denominators is zero .

29. Discuss the solution of $\begin{cases} x + a^2y + a^4z = a \\ x + b^2y + b^4z = b \\ x + c^2y + c^4z = c \end{cases}$, determining when the system has no solution, one solution, and infinitely many solutions.

30. Find the necessary and sufficient conditions for the simultaneous equations

$\begin{cases} x + ay + a^2z = 0 \\ x + by + b^2z = 0 \\ x + cy + c^2z = 0 \\ x + y + z = 0 \end{cases}$ to have solutions other than $x = y = z = 0$.

31. Discuss the solution of $\begin{cases} ax + by + z = 1 \\ x + aby + z = b \\ x + by + az = 1 \end{cases}$, determining when the system has no solution, and infinitely many solutions.

32. Solve the following equations by matrix methods:

(i) $\begin{cases} 4x - 3y + z = 11 \\ 2x + y - 4z = -1 \\ x + 2y - 2z = 1 \end{cases}$ **(ii)** $\begin{cases} x + 5y + 3z = 1 \\ 5x + y - z = 2 \\ x + 2y + z = 3 \end{cases}$.